**Perturbation Methods**

**Finding Roots of Equations**

Consider problem of solving

Generalize to the family of problems:   
Then expand the solution in powers of   
where is one of the solutions to the equation when   
You will get an infinite set of linear equation that can be solved to give each coefficient xi in the expansion. You must of course check that this Mclaurin series in does converge at , or whatever value of you want. This Mclaurin series would be exactly what you got if you expanded the actual general solution in a power series about . Note the need to specify , i.e. which one of the solutions the series will converge to.

Problems arise when x 0 is a double root. In such cases a convergent Mclaurin series may not exist about to give you one of the new roots - you would find that the coefficient equations demanded that some of them be inf or something. This would be reflected in the fact that if you tried to expand the actual general solution in a ML series in , you would find that it doesn't exist. Note the similarity of this problem with that of degenerate hermitian operators.

A singular perturbation is one that changes the order of the equation when becomes non zero. For instance:  
is singular because it changes from a linear to a quadratic equation. A similar thing will occur for ODE's and PDE's, when the term turns on a derivative higher than any of the nonperturbative terms.

Often times you will get one of the solutions to such an equation by expanding about the solution to the linear equation - again, you would see that that particular solution, if you solved the general equation, would have a convergent power series in , but the other one would not. So you miss it by demanding that it have convergent power series.

You can generalize these procedures to ones where you have multiple perturbative parameters, , and , etc., and to systems of more than one variable, etc.

Find the solution of the equation for large x .  
So we start with the following manipulations,  
Then I thought that I would try an iterative scheme,  
And further corrections would be obtained by further iterations,

But it turned out that the agreement between the actual solution and my perturbative solution collapsed after the second term in the series. So iterating in this fashion doesn't always work past the first term. The perturbative method gave the correct result. First we label the perturbative term with a lambda, and then expand our solution in powers of lambda. Finally reinsert our expansion into the equation and solve to each order in lambda.  
Plugging the expansion into the equation, I find,  
Then I expand and solve to each order (I'll just go up to order 2)

So we see that,  
So, as we can see, our iterative scheme worked only up to the first term in the expansion. The actual solution is,

Finding Eigenvalues, Eigenvectors of Matrices

You can, of course, also apply these ideas to matrices - finding inverses of a matrix, or finding the shift in eigenvalues, and eigenvectors.

For instance, if you have a non degenerate, hermitian matrix, A , and you know its eigenvalues, and eigenvectors, you can possibly find a perturbative expansion for the eigenvalues, and eigenvectors of .

So you want to find eigenvals, and eigenvecs of   
so let eigenvalues be

and solve equation to each order  
Of course you will have to use degenerate perturbation theory to handle the case when B lifts the degeneracies of whatever. And you will hope that the series is convergent in , even when it equals 1 , or whatever.

**Determining Solutions to ODEs, and PDEs**

Say you have and also some BC , like, say,

Then you will, as before, expand your solution in powers of , and solve to each order both the ODE , and the BC.

If you add a perturbation term like then you will probably not be able to get a perturbative solution since you have changed the order of the equation. Not only can you solve perturbed differential equations, but you can also solve perturbed .  
Suppose you have BC like:

![](data:application/octet-stream;base64,)

You can expand the perturbed BC in powers of .

Then expand in powers of , and solve the BC and PDE at each order.

**Finding Taylor Series, etc.**

These methods can also be applied to the more traditional problems:  
Find power series expansion for about

Now to get the coefficient an, you just take derivatives of with respect to , or equivalently, with respect to , since they occur in the combination , and set to 0 , and then divide by .

Note you could also allow negative powers of to get a Laurent series.

you can determine an expansion of in this fashion. To get a[-1] you would just multiply the expansion by , and take the limit as goes to zero. So the coefficient is as goes to 0 , which 1 . Obviously you would need to know what order pole you had, to be sure that all of your terms vanished when you multiplied by . This is how you get the residues in complex analysis. Note you could also determine the Mclaurin series for , and then just divide it by x. Though actually, I guess this wouldn't converge, because laurent series are only valid in an annular ring about 0 , in this case, such that is analytic in there, but isn't analytic in any anular ring.

**Singular PT**

Consider the equation,

Obviously one solution is near 1 , so let's perturbatively calculate that one.  
So we may in this way to determine the perturbed solution near 1 . But what about the other, as there always 2 solutions, for arbitrary eps. The reason we don't see the other solution is that it diverges as eps. goes to 0 , as we can see from the general formula,  
and regular PT assumes regular behavior near the limit. A way to find this other solution is to use what's called singular perturbation theory. What we do is rescale the unknown, , in terms of eps. so that as eps. goes to 0 , the other result remains finite. So we let  
so that eps. takes care of the divergence and allows to remain finite. Now plugging this into the equation:

which value of alpha should we choose? We choose the one which makes both exponents equal.



Now we apply regular PT again. We find that eps.^(-1) is our expansion parameter, but we'll just call it delta.

...  
So we note that the zeroth order equation has two roots, 0 and 1 . The 0 root is simply the one that we found earlier, and the 1 root is the one that we missed earlier.  


**Perturbation Theory**

Now we’re going to cover perturbation theory – both ordinary and singular perturbation theory. One subcase of this is the WKB approimation.

**Ordinary Perturbation Theory**

Perturbation methods are typically a global approximation scheme. The generally idea is to separate the problem into easy and hard parts,



append the ε to the hard part and assume a power series solution,



Then match powers of ε term and you develop a recursive series for the solution. If lucky, then the solution will have an infinite radius of convergence about ε = 0 and we can obtain the solution to our problem by summing up the series and setting ε = 1. But the power series so obtained may have only a finite radius of convergence about 0. Or it may not converge at all – i.e., it may be asymptotic so that we must truncate the series at a particular term to obtain the optimal approximation. Unfortunately, if ε is rather large then there may not be many terms in the truncated series and so the convergence may not be good. Nonetheless, there are methods to re-sum diverging asymptotic series so stay tuned.

If we’re unlucky though, we will find that the perturbative series cannot solve the problem, which indicates that the series doesn’t depend on ε in such a straightforward analytic manner. This can happen if for instance we append the ε to higher powers in an algebraic equation, or higher derivatives in an ODE. Generally, one can bet on the perturbative expansion in ε ‘powers’ to be non-analytic when the presence of the term fundamentally alters the behavior of the solution. Even still, we would be interested in such a series, arranged in order of increasing ‘powers’ of ε, if indeed ε is a small number.

**Example**

The problem:



may be written as:



And we’d be interested in solving the more general problem x(ε). Of course we would likely not be able to determine this either, but we might be able to get x(ε) for small ε, which would be a reasonable approximation to our original problem. So we’d assume that we could expand x(ε) in a power series and proceed accordingly. The ε = 0 value of x would be determined by:



and so using this as a basis we can write,



where x0 = one of the roots of the unperturbed problem. Then we have:



and these equations can be solved successively.

**Example**

Consider this problem,



We could start with



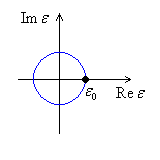
and demand,



Plugging this in, we’d find a bunch of recursive equations etc.

**Mathematical Structure of Perturbative Eigenvalue Problems**

But the validity of this approach depends on the analytic properties of x(ε). Once you construct the perturbative expansion of the eigenvectors yn(ε) and eigenvalues En(ε) or you may notice that the series blows up at ε = ε0. Additionally, the closer ε gets to ε0 the slower the series will converge. So if we were hoping to solve a problem for values of ε close to ε0, then we might run into difficulties.



This may be for a variety of reasons. It may be that at ε = ε0, the energy levels become degenerate (and perhaps later cross), or otherwise the problem becomes fundamentally different from the unperturbed problem, i.e., the problem becomes singular at ε = ε0. Mathematically this is because the formula for the En(ε) and yn(ε) have a pole or branch point somewhere on this radius. What we’d need to do then, basically, is factor out the non-analyticity from the perturbation expansion.

Singular Perturbation Theory

A perturbative problem becomes singular when:



So that basically setting ε = 0 is always a bad starting point. This is usually the case when the ε is multiplying a term of higher order than the terms present in the rest of the problem. It is also evident when the zeroth order term cannot satisfy the boundary conditions present in the problem. Consequently, usually.

**Example**

Consider:



which has solution:



as ε → 0, the solutions go to:



So the actual solution when ε = 0 and its behavior as ε increases can be captured with the perturbative expansion. But the other one cannot, as it blows up. But if we multiply the solutions by ε, then this would have a perturbative solution. So let’s look for a perturbative solution for y = εx. So then filling x = y/ε into the equation we have:



A perturbative expansion of the solutions here would yield y = ε, -1 to first non-vanishing order, which indeed corresponds to the actual first order solutions x = 1, -1/ε. So the trick is to determine the scale with which the largest non-perturbative solution blows up to ∞. How? Suppose we filled x = y/εp into the equation:



and so it is apparent that a perturbative solution would be possible for p = 1. Of course this makes sense since we know the general solution looks like that. How do we know that no other solutions are possible? Well p = 1 is a possibility because that would give an equation for a non-zero y, and the succeeding term(s) would be higher order. If tried to match the first and last term, then would have p = ½. But then the middle term would be of lower order and so this wouldn’t work. If tried to match last two terms, then we’d have p = 0. And this would work because the other term would be of higher order, but this is OK because p = 0 does start the series for another solution.

**Example**

Consider



When ε = 0 this equation has 3 roots, and when it doesn’t it has 6. So this is a singular perturbation problem and we would expect that the perturbation expansion of the missing roots contains fractional powers of ε. For the three roots that are present, this would present a valid perturbation problem and the setup can proceed as usual because the terms ε2x6 and εx4 would indeed be small compared to –x3 and 8 around the three unperturbed roots. But what about the ones that aren’t present when ε = 0? For these, the perturbed terms ε2x6 and εx4 are actually larger than the others because as ε → 0, the extra roots go to ∞ as 1/εn such that the perturbed terms actually blow up. If we can estimate about how much they blow up, then we can use the dominant balance method and work it out. Let’s assume x ~ 1/εp. Which would be the dominant terms?



Well the first term must be dominant in order to preserve the 6 roots. This means that 6p-2 > 4p – 1 → 2p > 1 → p > ½. This would mean that 4p-1 > 1 and 3p > 3/2. So the first and third terms are dominant. So we’d have:



just keeping the dominant terms. And so we have:



is how the roots must go. So what we can do is factor out this behavior from the solution. Let x = s/ε2/3 and plug into the equation. Then we have:



and then we could do PT on this problem. We would expect on this basis a series solution in the following form,



Note the fractional power expansion typical of singular perturbation theory.

**Example**

Consider the problem



This problem is a singular perturbation problem because the zeroth order problem has no solution.

**Example**

Another is the problem,



The perturbative series will be a sum of sines and cosines forever. But the actual solution goes into Airy equation territory and so is exponentially decreasing. The problem is that εx becomes large when x is large enough the perturbative series doesn’t globally apply. So this too is a singular perturbation theory when the whole half line is considered. We will need multiple scale analysis to address this problem.